

Cramér-Rao Lower Bound for DoA Estimation with RF Lens-Embedded Antenna Array

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Abstract—In this paper, we consider the Cramér-Rao lower bound (CRLB) for estimation of a lens-embedded antenna array with deterministic parameters. Unlike CRLB of uniform linear array (ULA), it is noted that CRLB for direction of arrival (DoA) of lens-embedded antenna array is dominated by not only angle but characteristics of lens. Derivation is based on the approximation that amplitude of received signal with lens is approximated to Gaussian function. We confirmed that parameters needed to design a lens can be derived by standard deviation of Gaussian, which represents characteristic of received signal, by simulation of beam propagation method. Well-designed lens antenna shows better performance than ULA in terms of estimating DoA. This is a useful derivation because, result can be the guideline for designing parameters of lens to satisfy certain purpose.

Index Terms—CRLB, lens antenna, DoA estimation.

I. INTRODUCTION

Reflecting the advent of a 5th generation (5G) communication, key technologies for next generation communication is under discussion. Extremely high data rate and low latency are some of the most notable changes in communication environment, to resolve those, massive multiple-input multiple-output (MIMO), millimeter-wave and densification are mentioned as core technologies in [1], [2]. Additionally, a novel approach to apply lens antenna to mmWave MIMO, which is commonly used for radar and satellite communication systems, was proposed in [3] based on advantages of high gain, narrow beamwidth and low sidelines in different directions.

In [4], it is proved that in ultra dense networks, beamforming utilizing location information based on line-of-sight (LOS) is superior compared to beamforming based on full band channel state information in terms of user throughput regardless of mobility of users, where the angles of departure and arrival is assumed to be given with respect to local coordinate systems, however, angles are obtained by acquisition of direction and distance information which is estimated by direction of departure (DoD), direction of arrival (DoA), time of arrival (ToA) and time of departure (ToD) in practice. The location information is one of the core elements for 5G communication in throughput boosting perspective.

With these overall superiorities of using DoA based beamforming, we focus on DoA estimation which constructs the

location information exploiting characteristic of received signal of lens antenna, whereas existing papers concerned about transmission and reception. In this paper, we compare error bound of DoA estimation lens antenna based on preceding research in [5] for uniform linear array (ULA). Cramér Rao lower bound (CRLB), well known bound on the variance of estimator of a deterministic parameter valid for unbiased estimator with additive Gaussian noise assumption, was used for criterion of comparison.

II. LENS CHARACTERISTIC

The characteristic of lens is defined by four parameters, f , D , T and ϵ_r , which the focal length, the aperture diameter, thickness and electric permittivity of lens respectively. Since our goal is to find the lower bound for error variance of DoA estimation, considering all of parameters is unnecessary and burden. All we need to derive the bound is the received signal of RF lens embedded antenna with expression of deterministic parameters.

The array response of lens antenna is determined by characteristic of lens and distance between lens and antenna. It was modeled as sinc function [6], where antenna elements are placed along the arc whose diameter is the same as the focal length. With lens assisted linear array, it is experimentally measured in [5], that the estimated coefficient of lens antenna response $\mathbf{a}(y; \phi)$ can be modeled as 1-Gaussian fitting model and each parameter of the fitted gaussian, $p(\phi)$, $q(\phi)$ and $r(\phi)$, are provided.

$$\mathbf{a}(y; \phi) = p(\phi) e^{-\left(\frac{y-q(\phi)}{r(\phi)}\right)^2} \quad (1)$$

We integrate $p(\phi)$, $q(\phi)$ and $r(\phi)$ to σ_c which is standard deviation of gaussian distribution denotes the amplitude of received signal and is a function. In array aspect, σ_c is a effective variable represents the characteristic of lens, thus we call it *curvature* of lens. The relationship between curvature and σ_c is reciprocal proportion. In other words, concentration of amplitude is maximized with large curvature and minimized with small curvature.

The relationship between parameters determines characteristic of lens and curvature is experimentally given in Table ?? . The relationship between parameters is experimentally confirmed by beam propagation method. For simplicity, we define $\mathbf{A}(\phi)$ is $N \times N$ diagonal matrix with (n, n) element follows Gaussian distribution with $\mathbf{A}(\phi)_{n,n} \sim \mathcal{N}\left(\frac{N-1}{\pi}\phi, \sigma_c\right)$. Then only mean changes linearly from end to end of array according

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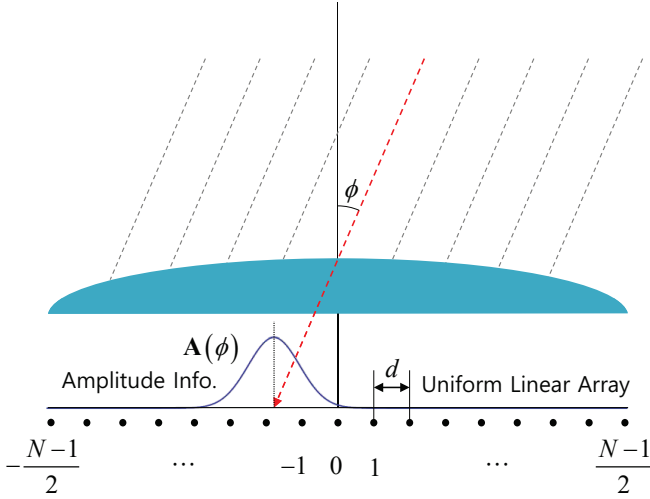


Fig. 1. System Model

to DoA, i.e., $q(\theta) = \phi(N-1)/\pi$ and $r(\phi) = \sigma_c^2$.

$$\mathbf{A}(\phi)_{n,n} = \sqrt{p_{lens}} \frac{1}{\sqrt{2\pi\sigma_c^2}} e^{-\frac{(n + \frac{N-1}{2}\phi)^2}{\sigma_c^2}} \quad (2)$$

Here, p_{lens} is a power normalizing factor which guarantees equal power gain between with and without lens. Define p_{lens} in DoA average sense to be fair.

$$\mathbb{E}_\phi \left[\sum \mathbf{A}(\phi)_{n,n}^2 \right] = N \quad (3)$$

III. SYSTEM MODEL

Let us consider an array antenna covered with lens depicted in Fig. 1. Array antenna has odd number of elements, N , equispaced by d with a signal coming from ϕ direction. The index of each element is set to be from $-N-1/2$ to $N-1/2$ to make the index of center 0. \mathbf{x} is $N \times 1$ column received signal vector, p is signal amplitude and b is phase without lens. $f(\sigma_c)$ is phase induced by characteristic of lens, which is deterministic value determined by phase transform function of the lens. $\mathbf{A}(\phi)$ is amplitude shape derived by lens. \mathbf{n} is additive white gaussian noise (AWGN) with noise variance σ_n .

$$\mathbf{x} = p\mathbf{A}(\phi) e^{j\{b+f(\sigma_c)\}} \mathbf{s}(\phi) + \mathbf{n} \quad (4)$$

$\mathbf{s}(\phi)$ represents the steering vector of the signal with direction ϕ . For convenience, we set phase reference is at the center of the array by using $z = e^{jkd \sin \phi}$ where k is time index.

$$\mathbf{s}(\phi) = \begin{bmatrix} z^{-(N-1)/2} \\ z^{-(N-3)/2} \\ \vdots \\ 1 \\ \vdots \\ z^{(N-3)/2} \\ z^{(N-1)/2} \end{bmatrix} \quad (5)$$

IV. CRLB ANALYSIS

In this section, we derive the bound of minimum error variance for estimating direction of arrival in terms of CRLB. The minimum error variance of any unbiased estimator is given by the CRLB in [7].

Lemma 1 (Cramér Rao Lower Bound). *Suppose θ is an unknown deterministic parameter estimated by measurements x with given probability density function $f(x|\theta)$. Then the variance of unbiased estimator $\hat{\theta}$ is bounded by the reciprocal of the Fisher information $I(\theta)$.*

$$\text{var}(\hat{\theta}) \geq \frac{1}{I(\theta)} \quad (6)$$

The Fisher information is given as follows:

$$I(\theta) = -\mathbb{E} \left[\frac{\partial^2 \ln(f(x|\theta))}{\partial \theta^2} \right] \quad (7)$$

A. CRLB for DoA Estimation of Linear Array

$\mathbf{A}(\phi)$ becomes identity matrix without lens. Broadly it is called ULA, and CRLB of it is derived in

$$\text{CRLB}_{no \text{ lens}}(\phi) = \frac{6\sigma_n^2}{p^2 N (N^2 - 1) k^2 d^2 \cos^2 \phi} \quad (8)$$

B. CRLB Analysis for DoA Estimation with Lens-Assisted Antenna Array

We define three parameters into a vector, $\theta = [p \ b \ \phi]$, includes amplitude, phase and direction of arrival. Note that ϕ is a parameter estimated from the received data. By setting $\mathbf{v} = p\mathbf{A}(\phi) e^{j\{b+f(\sigma_c)\}} \mathbf{s}(\phi)$, the probability density function given parameter vector θ is represented as follows.

$$f_{\mathbf{x}}(\mathbf{x}|\theta) = C e^{-(\mathbf{x}-\mathbf{v})^H \mathbf{R}^{-1} (\mathbf{x}-\mathbf{v})} \quad (9)$$

Where $\mathbf{R} = \sigma_n^2 \mathbf{I}$ and C is a normalization constant. The log-likelihood of (9) without additional constant is

$$g(\theta) = \frac{1}{\sigma_n^2} \left[p e^{-j\{b+f(\sigma_c)\}} \mathbf{s}^H(\phi) \mathbf{A}(\phi) \mathbf{x} + p e^{j\{b+f(\sigma_c)\}} \mathbf{x}^H \mathbf{s}(\phi) \mathbf{A}(\phi) - p^2 \mathbf{s}^H(\phi) \mathbf{A}^2(\phi) \mathbf{s}(\phi) \right] \quad (10)$$

Due to the fact that not all of off-diagonal terms in the Fisher information matrix are zero, we should calculate each

element of fisher information matrix which is,

$$\begin{aligned}
E \left[\frac{\partial^2 g}{\partial p^2} \right] &= -\frac{2}{\sigma_n^2} \mathbf{s}^H(\phi) \mathbf{A}^2(\phi) \mathbf{s}(\phi) \\
E \left[\frac{\partial^2 g}{\partial b^2} \right] &= -\frac{2p^2}{\sigma_n^2} \mathbf{s}^H(\phi) \mathbf{A}^2(\phi) \mathbf{s}(\phi) \\
E \left[\frac{\partial^2 g}{\partial \phi^2} \right] &= -\frac{2p^2}{\sigma_n^2} \frac{4(N-1)^2}{\pi^2 \sigma_c^4} \mathbf{s}^H(\phi) \mathbf{C}^2(\phi) \mathbf{A}^2(\phi) \mathbf{s}(\phi) \\
&\quad - \frac{2p^2}{\sigma_n^2} \mathbf{s}_1^H(\phi) \mathbf{A}^2(\phi) \mathbf{s}_1(\phi) \\
E \left[\frac{\partial^2 g}{\partial p \partial b} \right] &= 0 \\
E \left[\frac{\partial^2 g}{\partial b \partial \phi} \right] &= \frac{2jp^2}{\sigma_n^2} \mathbf{s}^H(\phi) \mathbf{A}^2(\phi) \mathbf{s}_1(\phi) \\
E \left[\frac{\partial^2 g}{\partial \phi \partial p} \right] &= \frac{4p(N-1)}{\pi \sigma_n^2 \sigma_c^2} \mathbf{s}^H(\phi) \mathbf{C}(\phi) \mathbf{A}^2(\phi) \mathbf{s}(\phi)
\end{aligned} \tag{11}$$

where \mathbf{s}_1 is partial derivative of \mathbf{s} , $\mathbf{s}_1(\phi) = \partial \mathbf{s}(\phi) / \partial \phi$. $\mathbf{C}(\phi)_{n,n} = n + \frac{N-1}{\pi} \phi$ is defined for simple expression of $\partial \mathbf{A} / \partial \phi$. For simplicity, (12) is used.

$$\begin{aligned}
D(\phi) &= \frac{p_{lens}}{2\pi \sigma_c^2} \sum_{n=-(N-1)/2}^{(N-1)/2} e^{-\frac{2(n+\frac{N-1}{\pi}\phi)^2}{\sigma_c^2}} \\
D_1(\phi) &= \frac{p_{lens}}{2\pi \sigma_c^2} \sum_{n=-(N-1)/2}^{(N-1)/2} n e^{-\frac{2(n+\frac{N-1}{\pi}\phi)^2}{\sigma_c^2}} \\
D_2(\phi) &= \frac{p_{lens}}{2\pi \sigma_c^2} \sum_{n=-(N-1)/2}^{(N-1)/2} n^2 e^{-\frac{2(n+\frac{N-1}{\pi}\phi)^2}{\sigma_c^2}} \\
D_{frac}(\phi) &= \frac{D(\phi)}{D(\phi) D_2(\phi) - D_1(\phi) D_1(\phi)}
\end{aligned} \tag{12}$$

D_{frac} exists always and is finite since $D(\phi) D_2(\phi) > D_1(\phi) D_1(\phi)$. Proof is provided in Appendix. Thus, determinant of the Fisher information matrix $\mathbf{J}(\theta)$ which is nonzero always exists and can be derived based on (12).

$$\det \{ \mathbf{J}(\theta) \} = \frac{8p^4}{\sigma_n^6} D(\phi) [D_1(\phi) D_1(\phi) - D(\phi) D_2(\phi)] \left[\frac{4(N-1)^2}{\pi^2 \sigma_c^4} + k^2 d^2 \cos^2 \phi \right] \tag{13}$$

The CRLB for DoA estimation with lens-assisted array antenna always exists as follows.

Theorem 1 (CRLB for DoA estimation of lens antenna). *The error variance for the DoA estimation problem of lens antenna is lower bounded by $CRLB_{lens}(\phi)$*

$$CRLB_{with\ lens}(\phi) = \frac{D_{frac}(\phi) \sigma_n^2}{2p^2 \left[\frac{4(N-1)^2}{\pi^2 \sigma_c^4} + k^2 d^2 \cos^2 \phi \right]} \tag{14}$$

By definition of p_{lens} and $D(\phi)$, $D(\phi) \approx N$ is reasonable approximation since it is sum of received power. For large

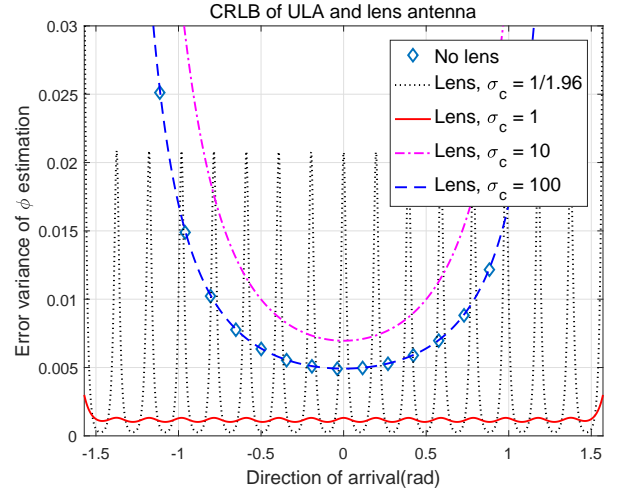


Fig. 2. CRLB comparison with lens antenna and ULA

σ_c , i.e., when curvature of lens is small, $\frac{4(N-1)^2}{\pi^2 \sigma_c^4} \approx 0$ and $e^{-\frac{2(n+\frac{N-1}{\pi}\phi)^2}{\sigma_c^2}} \approx 1$. Then,

$$\begin{aligned}
D_1(\phi) &\approx \sum_{n=-(N-1)/2}^{(N-1)/2} n \frac{p_{lens}}{2\pi \sigma_c^2} \approx 0 \\
D_2(\phi) &\approx \sum_{n=-(N-1)/2}^{(N-1)/2} n^2 \frac{p_{lens}}{2\pi \sigma_c^2} \approx \sum_{n=-(N-1)/2}^{(N-1)/2} n^2 \\
&= \frac{N(N^2-1)}{12}
\end{aligned} \tag{15}$$

Therefore (14) with large σ_c can be approximated

$$CRLB_{with\ lens}(\phi) \approx \frac{6\sigma_n^2}{p^2 N (N^2-1) k^2 d^2 \cos^2 \phi} \tag{16}$$

(16) is identical to (8). It implies that DoA estimation based on lens antenna with small curvature will show similar performance without lens which is intuitively understandable.

C. Graphical Description of CRLB

In this section, we compared CRLB of antenna with and without lens according to result of previous chapter. The number of antenna is 17 and σ_c is selected from 1/1.96 from 100 to represent various curvature of lens.

1) *Small curvature*: By setting σ_c to 100 and 10, lens with small curvature is modeled. Since power is normalized, the performance of it is expected to be similar to that without lens, and it is shown by the simulation, which also confirms that derivation of CRLB for DoA estimation with lens is proper. When $\sigma_c = 10$, the bound goes up which means received SNR is not properly focused to boost DoA estimation.

2) *Proper curvature*: With proper curvature of lens, estimation error variance decreases greatly. This implies that the optimum curvature in aspect of DoA estimation can exist.

3) *Large curvature*: The standard deviation of large curvature is set to be 1/1.96 to depict that power from center of maximum amplitude to d length for both sides of antenna is 95% of total received power. Due to the system model

that mean of Gaussian slides linearly according to direction, 95% of total power is received by one or two elements only. Intuitively performance will be degraded more when almost whole power is received by a element. It is well shown in Fig. 2. When mean of Gaussian is nearby the antenna element, i.e., almost whole power is concentrated to a antenna, bound goes up through the case without lens, however, error bound is extremely small in case of mean positioning between two elements on the other hand.

V. CONCLUSION

In this letter, we investigated a CRLB of lens embedded antenna array with proper model based on experimental results of previous research. Our results demonstrate that lens antenna has potential to show better performance in an aspect of DoA estimation than without lens. The algorithm achieves the bound should be studied which calls for further future research.

VI. APPENDIX

Proof: Here we define $f(p, m)$ for simple expression of $D(\phi) D_2(\phi) > D_1(\phi) D_1(\phi)$.

$$f(p, m) = \sum_{k=-p}^p k^m a_k, \text{ where } a_k = e^{-\frac{2(k+n)^2}{\sigma_c^2}} \quad (17)$$

Then (12) can be represented if $n = \phi(N-1)/\pi$. Note that $D(\phi) D_2(\phi) > D_1(\phi) D_1(\phi)$ is true when $p = 1$. That is,

$$f(1, 2) f(1, 0) - \{f(1, 1)\}^2 = 4a_{-1}a_1 + a_{-1}a_0 + a_0a_1 > 0 \quad (18)$$

Suppose the inequality holds when $p = \ell$, which is $f(\ell, 2) f(\ell, 0) - \{f(\ell, 1)\}^2 > 0$. Then, $p = \ell + 1$ case can be shown in terms of (19)

$$\begin{aligned} f(\ell+1, 0) &= a_{-(\ell+1)} + f(\ell, 0) + a_{(\ell+1)} \\ f(\ell+1, 1) &= -(\ell+1)a_{-(\ell+1)} + f(\ell, 1) + (\ell+1)a_{(\ell+1)} \\ f(\ell+1, 2) &= (\ell+1)^2 a_{-(\ell+1)} + f(\ell, 2) + (\ell+1)^2 a_{(\ell+1)} \end{aligned} \quad (19)$$

The inequality when $p = \ell + 1$ can be expressed,

$$\begin{aligned} f(\ell+1, 2) f(\ell+1, 0) - \{f(\ell+1, 1)\}^2 &= f(\ell, 2) f(\ell, 0) - f(\ell, 1) f(\ell, 1) + 4(\ell+1)^2 a_{(\ell+1)} a_{-(\ell+1)} \\ &+ \left[f(\ell, 2) + 2(\ell+1) f(\ell, 1) + (\ell+1)^2 f(\ell, 0) \right] a_{-(\ell+1)} \\ &+ \left[f(\ell, 2) - 2(\ell+1) f(\ell, 1) + (\ell+1)^2 f(\ell, 0) \right] a_{(\ell+1)} \end{aligned} \quad (20)$$

If $f(\ell, 1) > 0$, the mean of Gaussian is negative number so that $a_{(\ell+1)} > a_{-(\ell+1)}$ holds. Thus we can express $f(\ell, 1)$ as $\sqrt{f(\ell, 2) f(\ell, 0)} - \varepsilon$ with some positive constant ε .

$$\begin{aligned} f(\ell+1, 2) f(\ell+1, 0) - \{f(\ell+1, 1)\}^2 &= [f(\ell, 2) f(\ell, 0) - f(\ell, 1) f(\ell, 1)] + 4(\ell+1)^2 a_{(\ell+1)} a_{-(\ell+1)} \\ &+ \left[\sqrt{f(\ell, 2) f(\ell, 0)} + (\ell+1) \sqrt{f(\ell, 0)} \right]^2 a_{-(\ell+1)} \\ &+ \left[\sqrt{f(\ell, 2) f(\ell, 0)} - (\ell+1) \sqrt{f(\ell, 0)} \right]^2 a_{(\ell+1)} \\ &+ 2(\ell+1) \varepsilon \{a_{(\ell+1)} - a_{-(\ell+1)}\} \end{aligned} \quad (21)$$

Since (21) is sum of positive terms, it is greater than zero. In case of $f(\ell, 1) < 0$, same process with opposite sign shows inequality holds. ■

REFERENCES

- [1] J. G. Andrews, S. Buzzi, W. Choi, S. V. Hanly, A. Lozano, A. C. K. Soong, and J. C. Zhang, "What Will 5G Be?" *IEEE Journal on Selected Areas in Communications*, vol. 32, no. 6, pp. 1065–1082, 2014.
- [2] F. Boccardi, R. Heath, A. Lozano, T. Marzetta, and P. Popovski, "Five disruptive technology directions for 5G," *IEEE Communications Magazine*, vol. 52, no. 2, pp. 74–80, 2014.
- [3] Y. Zeng, R. Zhang, and Z. N. Chen, "Electromagnetic Lens-Focusing Antenna Enabled Massive MIMO: Performance Improvement and Cost Reduction," *IEEE Journal on Selected Areas in Communications*, vol. 32, no. 6, pp. 1194–1206, 2014.
- [4] P. Kela, M. Costa, J. Turkka, M. Koivisto, and J. Werner, "Location based beamforming in 5G ultra-dense networks," in *2016 84th Vehicular Technology Conference (VTC Fall)*. IEEE, Sep. 2016.
- [5] T. Kwon, Y.-G. Lim, B.-W. Min, and C.-B. Chae, "RF Lens-Embedded Massive MIMO Systems: Fabrication Issues and Codebook Design," *IEEE Transactions on Microwave Theory and Techniques*, vol. 64, no. 7, pp. 2256–2271, 2016.
- [6] Y. Zeng and R. Zhang, "Millimeter Wave MIMO With Lens Antenna Array: A New Path Division Multiplexing Paradigm," *IEEE Transactions on Communications*, vol. 64, no. 4, pp. 1557–1571, 2016.
- [7] S. M. Kay, *Fundamentals of Statistical Signal Processing: Practical algorithm development*. Pearson Education, 2013.